Ramsey Tilings

Lanchao Wang

Nanjing University and IBS

Joint work with Hong Liu, Maksim Turevskii and Zhifei Yan

November 2026

Outline

We write mH for an H-tiling consisting of m vertex-disjoint copies of H.

In this talk, we study the maximum size of a monochromatic H-tiling guaranteed by every r-edge-colouring of a host graph G.

1. Complete host graphs
2. Random host graphs
3. Minimum-degree host graphs
4. Hypergraphs

Complete host graphs

Triangle tilings

$R_r(mH)$ is the least N such that every r-edge-colouring of K_N contains mH in one colour.

Theorem (Burr, Erdős and Spencer, 1975)

For every integer $m \geqslant 2$,

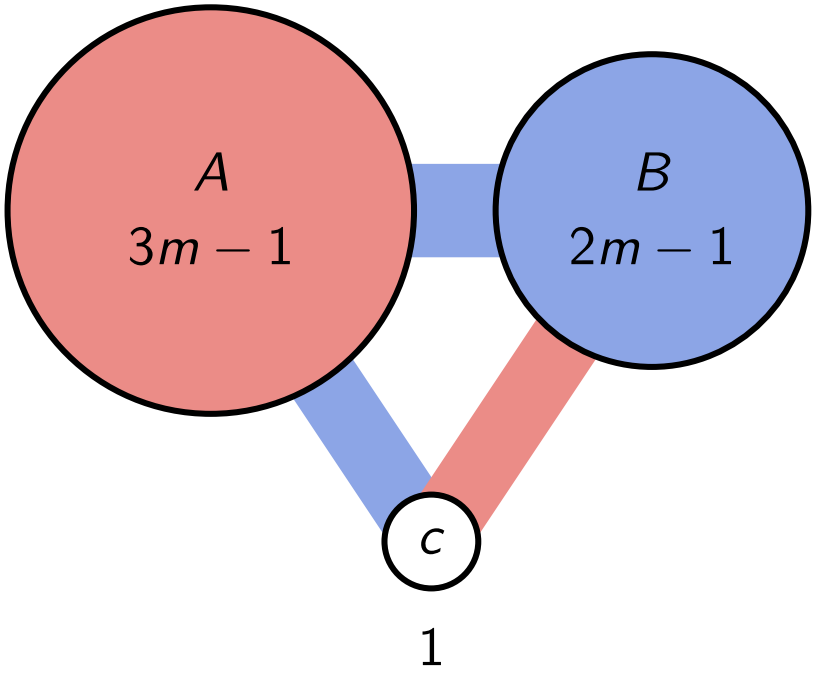
$$R_2(mK_3) = 5m.$$

A trivial upper bound: $R_2(mK_3) \leqslant 6m$.

Consider any red-blue colouring of K_{6m}. Since $R_2(K_3) = 6$, we can repeatedly remove a monochromatic triangle until we have $2m - 1$ vertex-disjoint triangles. At least m of them have the same colour.

Triangle tilings: the extremal colouring

A colouring of K_{5m-1} with no monochromatic mK_3.



Red tilings $\mathcal{T}_R$

Every red triangle lies in A.

$$|\mathcal{T}_R| \leqslant \left\lfloor \frac{|A|}{3} \right\rfloor = m - 1.$$

Blue tilings $\mathcal{T}_B$

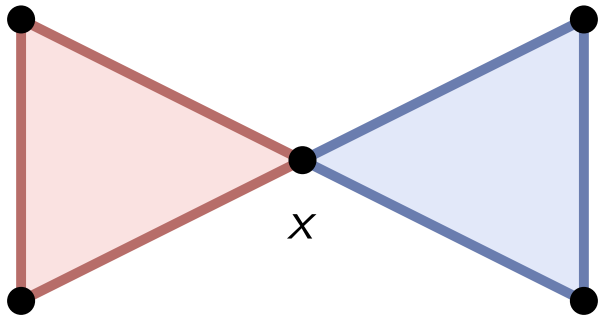
Each triangle meets B in at least two vertices.

$$|\mathcal{T}_B| \leqslant \left\lfloor \frac{|B|}{2} \right\rfloor = m - 1.$$

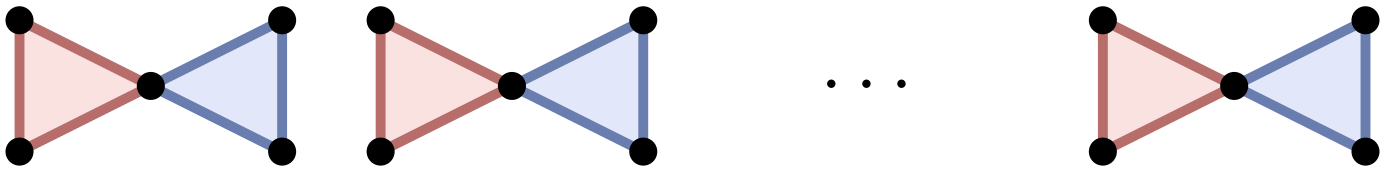
$$R_2(mK_3) \geqslant 5m.$$

Upper bound: the bowtie argument

A red triangle and a blue triangle sharing one vertex.



Repeatedly remove vertex-disjoint bowties. The critical case is when the $5m$ vertices split into m disjoint bowties.



m vertex-disjoint bowties

Choosing the same colour in every bowtie gives a monochromatic mK_3.

General graphs

The same approach extends to arbitrary graphs H. For general H, the bowtie consists of a red and a blue copy of H sharing a maximum independent set.

Theorem (Burr, Erdős and Spencer, 1975)

Fix a graph H without isolated vertices. There exists a constant C_H such that, for all sufficiently large m,

$$R_2(mH) = \Big(2v(H) - \alpha(H)\Big)m + C_H.$$

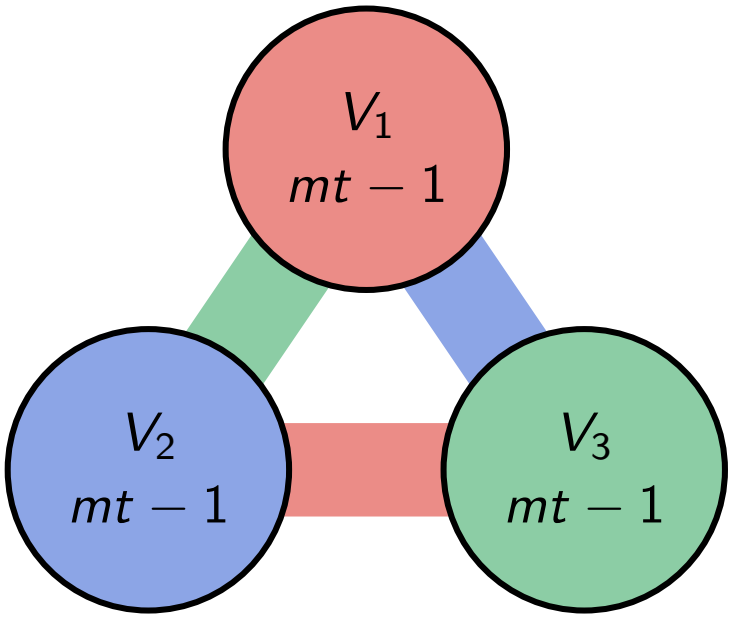
Here $\alpha(H)$ is the independence number of H.

However, the bowtie argument only works for two colours.

Multicolors: non-bipartite tilings

Theorem (Lorimer and Segedin, 1978)

Fix $r \geqslant 3$ and $t \geqslant 3$. Then $R_r(mK_t) = rtm + O_{r,t}(1)$.



For $r = 3$, take three clusters, each of size $mt - 1$.

Every red copy of K_t lies entirely inside V_1.

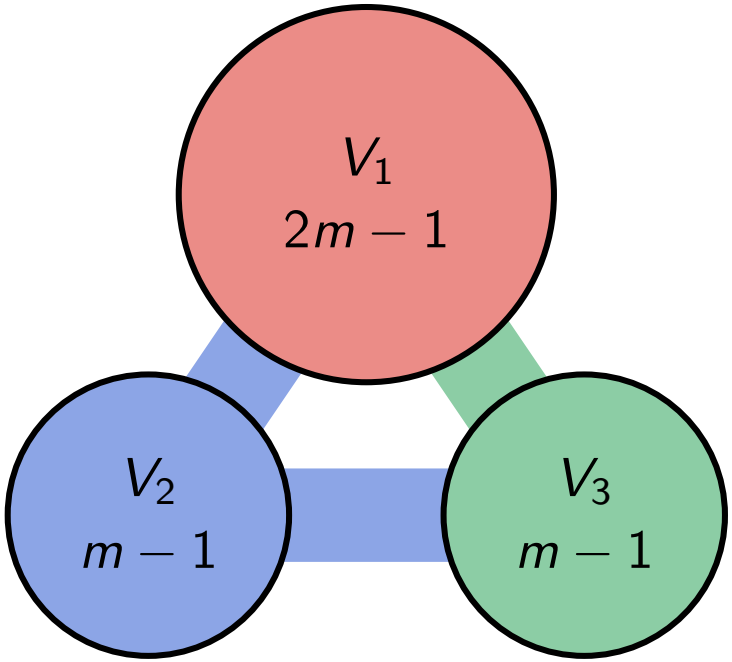
Hence $R_3(mK_t) \geqslant 3mt - 2$.

As observed by Gyárfás, Sárközy and Selkow, the same holds for every connected non-bipartite graph H: $R_r(mH) = r\,v(H)m + O_{r,H}(1)$.

Multicolors: matchings

Theorem (Cockayne and Lorimer, 1975)

For every $r \geqslant 2$ and $m \geqslant 1$, $R_r(mK_2) = (r+1)m - r + 1$.



For $r = 3$, take three clusters.

$R_3(mK_2) \geqslant 4m - 2$.

Prior to our work, for $r \geqslant 3$, $R_r(mH)$ is unknown for general connected bipartite graphs H beyond the matching case.

It is also open for general disconnected graphs H.

Ramsey tilings: our main theorem

Theorem (with Liu, Turevskii and Yan, 2026+)

Let $r \geqslant 2$ and let H be a graph. There exist a constant ρ, a positive integer T, and integers $b_0, \ldots, b_{T-1}$, all depending only on r and H, such that ρ is computable from finitely many linear programs and

$$R_r((Tk+j)H) = \rho Tk + b_j$$

for every $j \in \{0, \ldots, T-1\}$ and all sufficiently large k.

Corollary

For every $r \geqslant 2$ and every graph H,

$$R_r(mH) = \rho m + O_{r,H}(1),$$

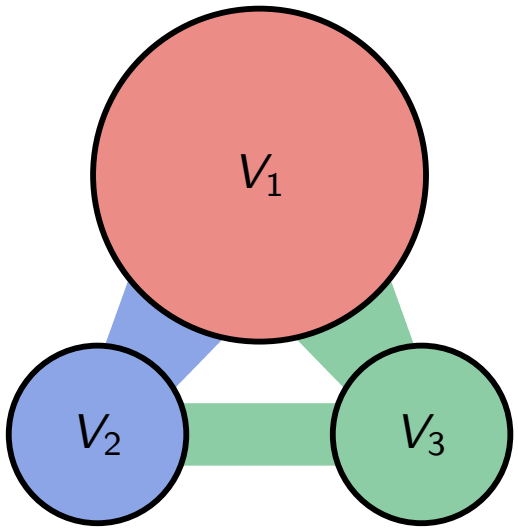
where ρ is computable from finitely many linear programs.

Example: $K_{a,b}$ with three colours

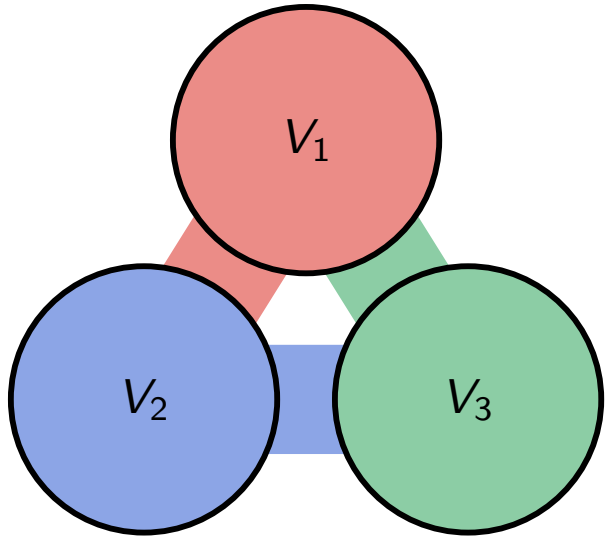
Theorem (with Liu, Turevskii and Yan, 2026$^+$)

For $b \geqslant a \geqslant 1$,

$$R_3(mK_{a,b}) = \begin{cases} (3a+b)m + O(1), & a \leqslant b \leqslant 3a, \\ \dfrac{3(a+b)}{2}m + O(1), & b \geqslant 3a. \end{cases}$$



$b \leqslant 3a$



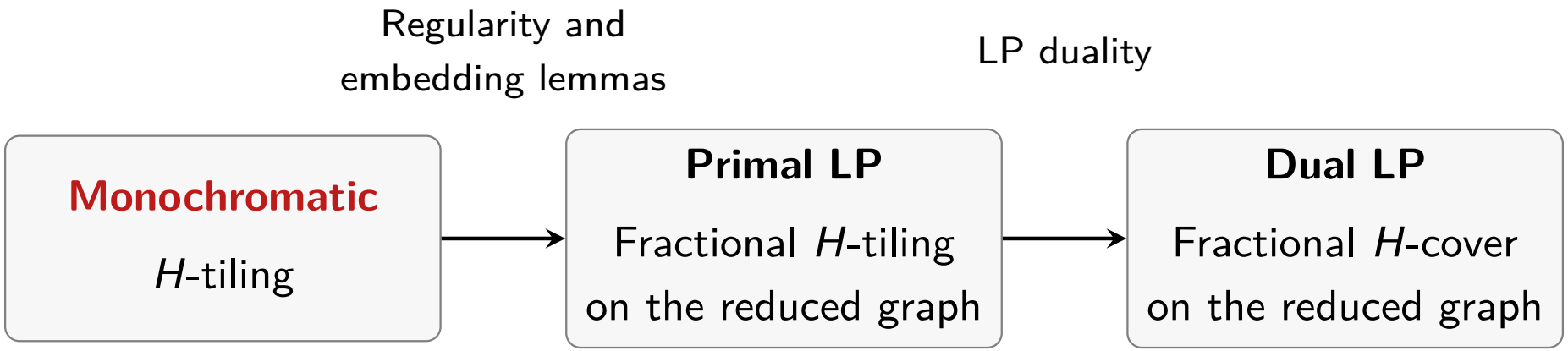
$b \geqslant 3a$

Proof strategy

1. **Regularity.** Fix any coloring. Apply the multicolour regularity lemma to partition the graph into M clusters.

2. **LP duality.** Encode monochromatic H-tilings by linear programs, assigning an r-dimensional weight vector to each cluster via duality.

Thus, the lower-bound for monochromatic H-tilings reduces to estimating the sum of M vectors in $\mathbb{R}^r$.



Proof strategy

Let $\mathcal{W}$ denote the multiset of these M weight vectors. We obtain

$$\text{monochromatic } H\text{-tiling number} \geqslant \sum_{w \in \mathcal{W}} w - o(n).$$

3. **Convex geometry.** Move the average weight vector to the boundary of its convex hull. The resulting point is a convex combination of at most r vectors.

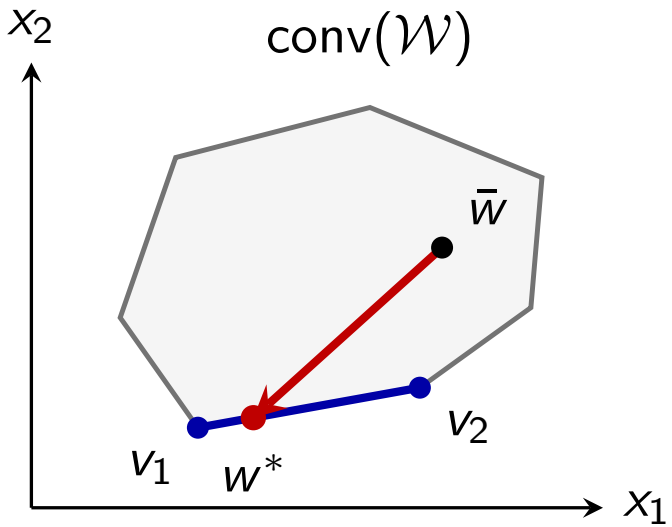


Illustration for $r = 2$.

x_1:red, x_2:blue

Move in the direction $(-1, -1)$.

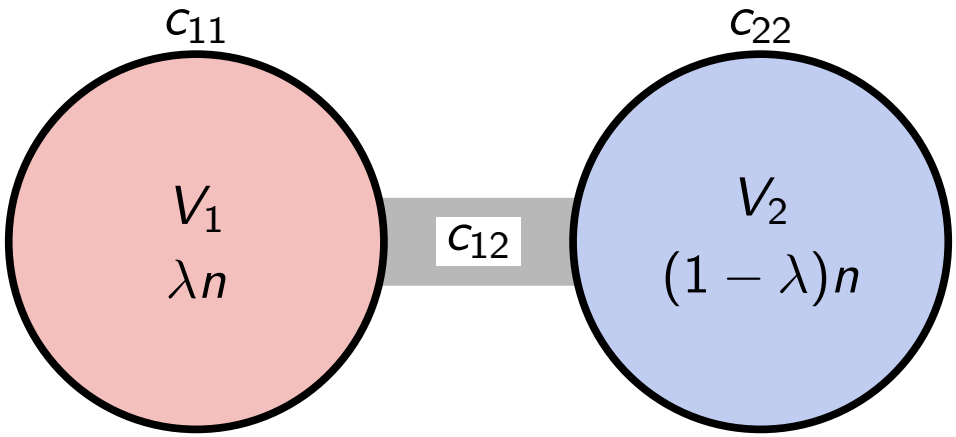
The boundary point

$$w^* = \lambda v_1 + (1 - \lambda) v_2, \qquad 0 \leqslant \lambda \leqslant 1.$$

Proof strategy

4. **Finite-complexity colouring.** Use these at most r vectors and their convex coefficients to construct an r-colouring with at most r vertex classes.

For $r = 2$: From $w^* = \lambda v_1 + (1-\lambda)v_2$, partition $V(K_n) = V_1 \cup V_2$.



Choose colours $c_{11}, c_{12}, c_{22} \in \{1, 2\}$. Every such colouring is determined by λ and c_{11}, c_{12}, c_{22}, giving only finitely many templates.

monochromatic H-tiling $\geqslant$ minimizing over these finitely templates $- o(n)$

Proof strategy

5. **Exactness.** Combine the finite-complexity reduction with a stability-type argument to obtain an exact result.

Balanced blocks. For a suitable positive integer T, every optimal strip-colouring template admits an r-edge-coloured complete graph on ρT vertices containing a monochromatic TH in every colour.

A stability argument shows that every sufficiently large near-extremal r-edge-colouring of K_n contains such a balanced block.

This gives the exact recurrence

$$R_r((m+T)H) = R_r(mH) + \rho T.$$

Related work

Theorem (Freschi, Martin and Treglown, 2026+)

Independently, they proved that

$$R_r(mH) = \rho_{r,H} m + O_{r,H}(1)$$

in the following cases:

- $r = 3$, *for every graph* H;
- $r \geqslant 4$, *whenever* $\chi(H) \geqslant r$.

The block used in our stability argument is inspired by them.

Random host graphs

From complete to random graphs

What happens if we replace K_n by $G \sim G(n,p)$?

Let $\mathrm{Rt}_r(H;G)$ denote the maximum number of vertex-disjoint copies of H in one colour guaranteed by every r-colouring of G.

Equivalently,

$$R_r(mH) = \min\{n : \mathrm{Rt}_r(H;K_n) \geqslant m\}.$$

Random graphs: two colours

Two independent works (2026+):

- Aragão, Cheng, Filipe, Miyazaki, Peng and Yan;
- Freschi, Martin and Treglown.

Theorem (Random Burr–Erdős–Spencer theorem)

Let H be a fixed graph without isolated vertices. If $G \sim G(n,p)$ with $p \gg n^{-1/m_2^*(H)}$, then w.h.p.

$$\mathrm{Rt}_2(H;G) = \mathrm{Rt}_2(H;K_n) - o(n).$$

Here $m_2^*(H)$ is the parameter such that $G(n,p)$ is w.h.p. 2-Ramsey for H whenever $p \gg n^{-1/m_2^*(H)}$.

Random graphs: multicolour matchings

Theorem (Gishboliner, Krivelevich and Michaeli, RSA, 2022)

Fix $r \geqslant 2$. If $G \sim G(n,p)$ with $p \gg 1/n$, then w.h.p.

$$\mathrm{Rt}_r(K_2; G) = \mathrm{Rt}_r(K_2; K_n) - o(n).$$

Random graphs: arbitrary colours

Theorem (with Liu, Turevskii and Yan, 2026$^+$)

Fix $r \geqslant 2$ and a graph H with at least one edge. If $G \sim G(n,p)$ with

$$p \gg n^{-1/m_2^*(H)},$$

then w.h.p.

$$\mathrm{Rt}_r(H; G) = \mathrm{Rt}_r(H; K_n) - o(n).$$

The proof is essentially the same as for complete graphs, except that the regularity lemma is replaced by its version for random graphs. The final stability argument is no longer needed.

Minimum-degree host graphs

Minimum degree: triangle tilings

We now consider arbitrary n-vertex graphs G with large minimum degree $\delta(G)$.

Theorem (Balogh, Freschi and Treglown, EJC, 2026)

Every 2-edge-coloured n-vertex graph G satisfies

$$\mathrm{Rt}_2(K_3; G) \geqslant \begin{cases} \left\lfloor \dfrac{\delta(G)+1}{5} \right\rfloor, & \delta(G) \geqslant \frac{65n}{66}, \\ \left\lceil \dfrac{5\delta(G)-4n}{2} \right\rceil, & \frac{4n}{5} \leqslant \delta(G) \leqslant \frac{5n}{6}. \end{cases}$$

Both bounds are best possible.

The intermediate range

$$\frac{5n}{6} < \delta(G) < \frac{65n}{66}$$

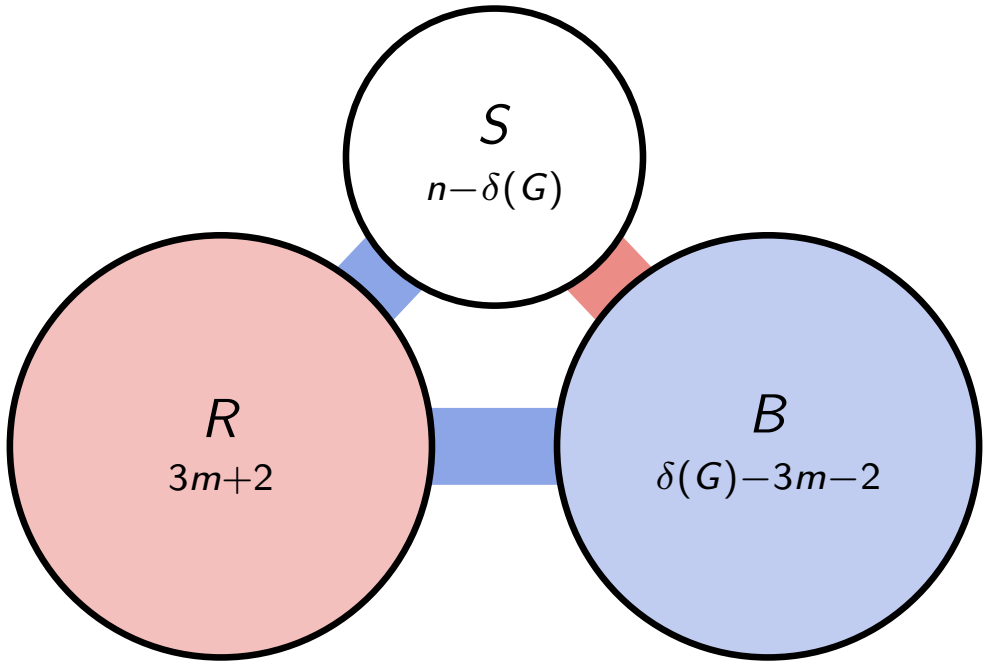
is not covered by the theorem.

First extremal example

Sharpness of

$$\mathrm{Rt}_2(K_3; G) \geqslant \left\lfloor \frac{\delta(G)+1}{5} \right\rfloor \qquad \text{when } \delta(G) \geqslant \frac{65n}{66}.$$

Let $m = \lfloor (\delta(G)+1)/5 \rfloor$.



Minimum degree: triangle tilings

Question (Balogh, Freschi and Treglown, 2026)

Let $n \geqslant 25$ and let G be a 2-edge-coloured n-vertex graph. Is it true that

$$\mathrm{Rt}_2(K_3; G) \geqslant \begin{cases} \left\lfloor \dfrac{\delta(G)+1}{5} \right\rfloor, & \delta(G) \geqslant \frac{15n}{17}, \\ \left\lfloor \dfrac{4\delta(G)-3n+1}{3} \right\rfloor, & \frac{6n}{7} \leqslant \delta(G) \leqslant \frac{15n}{17}, \\ \left\lceil \dfrac{5\delta(G)-4n}{2} \right\rceil, & \frac{4n}{5} \leqslant \delta(G) \leqslant \frac{6n}{7}? \end{cases}$$

All three bounds are best possible, as shown by their extremal constructions.

Minimum degree: multicolour matchings

Theorem (Omidi, Raeisi and Rahimi, EJC, 2018)

Let $r \geqslant 2$, and let G be an n-vertex graph. Then

$$\mathrm{Rt}_r(K_2; G) \geqslant \begin{cases} \left\lceil \dfrac{\delta(G)}{r} \right\rceil, & 0 \leqslant \delta(G) < \dfrac{r}{r+1} n, \\ \left\lfloor \dfrac{n+r-1}{r+1} \right\rfloor, & \dfrac{r}{r+1} n \leqslant \delta(G) \leqslant n-1. \end{cases}$$

Both bounds are best possible.

In particular, if

$$\delta(G) \geqslant \frac{r}{r+1} n,$$

then

$$\mathrm{Rt}_r(K_2; G) \geqslant \mathrm{Rt}_r(K_2; K_n) = \left\lfloor \frac{n+r-1}{r+1} \right\rfloor.$$

Minimum degree: bipartite tilings

Theorem (with Wang and Yu, 2026+)

Let $1 \leqslant a \leqslant b$ and $0 \leqslant \delta \leqslant 1$. Every 2-edge-coloured n-vertex graph G with $\delta(G) \geqslant \delta n - 1$ satisfies

$$\mathrm{Rt}_2(K_{a,b}; G) \geqslant \begin{cases} \dfrac{\delta n}{2a} - o(n), & 0 \leqslant \delta \leqslant \frac{a}{a+b}, \\ \dfrac{n}{2(a+b)} - o(n), & \frac{a}{a+b} \leqslant \delta \leqslant \frac{1}{2}, \\ \dfrac{\delta n}{a+b} - o(n), & \frac{1}{2} \leqslant \delta \leqslant \frac{a+b}{2a+b}, \\ \dfrac{n}{2a+b} - o(n), & \frac{a+b}{2a+b} \leqslant \delta \leqslant 1. \end{cases}$$

All four bounds are asymptotically best possible.

For arbitrary $r \geqslant 3$, we also obtain an analogous result via LP.

An obstruction to an $O(1)$ error term

For complete host graphs, we can achieve an $O(1)$ error term.

Minimum-degree conditions, the $o(n)$ error term cannot in general be improved to $O(1)$.

Theorem (with Wang and Yu, 2026+)

For infinitely many n, there exists an n-vertex graph G with $\delta(G) \geqslant 2n/5 - 1$ such that

$$\mathrm{Rt}_3(K_{2,4}; G) \leqslant \frac{n}{15} - \Omega(\sqrt{n}).$$

Minimum-degree conditions allow us to insert sparse C_4-free bipartite graphs between clusters, leading to an $\Omega(\sqrt{n})$ error term.

Hypergraphs

The Alon–Frankl–Lovász theorem

Theorem (Alon, Frankl and Lovász, 1986)

For all integers $r, t \geqslant 2$ and $n \geqslant t$,

$$\mathrm{Rt}_r(K_t^{(t)}; K_n^{(t)}) = \left\lfloor \frac{n+r-1}{t+r-1} \right\rfloor.$$

- It determines the chromatic number of Kneser hypergraphs, resolving a conjecture of Erdős (1973).
- It is the first landmark combinatorial theorem proved using topology, which marked the beginning of topological combinatorics.

Proof Methods and Extensions of the AFL Theorem

- **Sarkaria (JCTB, 1990), Kriz (TAMS, 1992)**
  Equivariant topology and index theory.

- **Ziegler (Invent. Math., 2002)**
  Combinatorial proofs via $\mathbb{Z}_p$-Tucker-type lemmas.

- **Haviv (ACM TOCT, 2025)**
  A new proof via consensus division.

All rely on topological tools and study complete host hypergraphs.

Random hypergraphs: the AFL theorem

Theorem (Gishboliner, Glock, Michaeli and Sgueglia, CPC, 2026)

Fix $r, t \geqslant 2$. If $G \sim G^{(t)}(n,p)$ with $p \gg n^{-(t-1)}$, then w.h.p.

$$\mathrm{Rt}_r(K_t^{(t)}; G) = \mathrm{Rt}_r(K_t^{(t)}; K_n^{(t)}) - o(n).$$

Thus, the AFL theorem extends asymptotically to sparse random hypergraphs. Their proof uses the classical AFL theorem as a black box.

Prior to our work, no topology-free proof of the AFL theorem was known, nor was it known whether the AFL bound could be preserved under minimum-degree conditions.

An exact robust AFL theorem

Theorem (with Liu, Turevskii and Yan, 2026$^+$)

Fix $r, t \geqslant 2$. There exist $c = c_{r,t} > 0$ and n_0 such that for every $n \geqslant n_0$, every t-uniform hypergraph G on n vertices satisfying

$$\delta(G) \geqslant (1-c)\binom{n-1}{t-1}$$

has

$$\mathrm{Rt}_r(K_t^{(t)}; G) = \mathrm{Rt}_r(K_t^{(t)}; K_n^{(t)}) = \left\lfloor \frac{n+r-1}{t+r-1} \right\rfloor.$$

- The exact AFL bound is preserved even when the host hypergraph is not complete.
- Taking $G = K_n^{(t)}$ gives a topology-free proof of the exact AFL theorem for all sufficiently large n.

Proof strategy: robust AFL

Our proof follows essentially the same strategy as before:

$$\text{Regularity} \longrightarrow \text{LP duality} \longrightarrow \text{Convex geometry}.$$

The key difference lies in the exactness step:
We first exploit the rigidity of extremal dual LP solutions to show that the original colouring is close to an extremal template.

We then construct a matching in this template. Although some of its edges may be missing or have the wrong colour, these bad edges are sufficiently sparse locally.

Finally, a switching argument allows us to replace all bad edges, yielding the desired monochromatic matching.

Stability of the AFL theorem

Building on our framework, with further refinements, Kan, Lu and Yuan recently proved the following stability theorem.

Theorem (Kan, Lu and Yuan, 2026+)

Fix $r, t \geqslant 2$. Every r-edge-colouring of $K_n^{(t)}$ whose largest monochromatic matching has size at most

$$\mathrm{Rt}_r(K_t^{(t)}; K_n^{(t)}) + o(n)$$

differs from a threshold extremal colouring on only $o(n^t)$ edges.

Open problems

1. Sharp minimum-degree threshold for AFL.

Determine the smallest constant $d_{r,t}$ such that

$$\delta(G) \geqslant d_{r,t}\binom{n-1}{t-1} \quad \Longrightarrow \quad \mathrm{Rt}_r(K_t^{(t)}; G) = \mathrm{Rt}_r(K_t^{(t)}; K_n^{(t)})$$

for all sufficiently large n.

2. Hypergraph tilings beyond matchings.

Can our LP and convex-geometric framework determine

$$\mathrm{Rt}_r(\mathcal{H}; K_n^{(t)}) = (\beta_{r,\mathcal{H}} + o(1))n$$

for general t-uniform hypergraphs $\mathcal{H}$?

3. Minimum-degree Ramsey triangle tilings.

Determine $\mathrm{Rt}_2(K_3; G)$ in terms of $\delta(G)$, as asked by Balogh, Freschi and Treglown.

Thank you!